Boussinesq Approximation in Stratified Fluids

Learning Objectives:

1. List and explain the assumptions behind the classical equations of fluid dynamics
2. Identify and formulate the physical interpretation of the mathematical terms in solutions to fluid dynamics problems
3. Write and explain the governing equations for weakly-stratified fluids
4. Explain the physical interpretation of the buoyancy frequency

Topics/Outline:

1. Occurrence of stratification
2. Boussinesq approximation
3. Vorticity equation
4. Governing parameters in stratified flows

Reading:

Kundu, P.K. and Cohen, I.M. (2008). Fluid Mechanics, fourth edition. Academic Press: New York. QA901 .K86 2008, previous edition as electronic resource: QA901 .K86 2002eb
4.18

Turner, J.S. (1973). Buoyancy Effects in Fluids. Cambridge University Press: Cambridge, UK. QA911 .T85 1979
Scan Chapters 1, 2, and 4.
Chapter 1

Stratified Flows: Accounting for density variation.

Most ocean flows have some degree of stratification:

- Salinity gradients in estuaries.
- Lateral temperature gradients in estuaries, bays, and wetlands.
- Vertical temperature gradients near the ocean surface.
- Salinity and temperature gradients over deep ocean profiles.

Stratification leads to a vertical body force that retards vertical motion and can generate interfacial and internal waves. Progression of internal waves can cause boundary mixing far from the perturbation source and complicates the turbulence dynamics.

Stratification also alters the speed of sound, complicating acoustic oceanographic methods.

Hence, and understanding of the dynamic effects of stratification is critical for predicting and understanding hydromechanics in the oceans.

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Governing Equations:

Assumptions:

1. All assumptions for Navier-Stokes equations.
2. Incompressible fluid
3. Non-diffusive (no mixing)

(2 and 3) $\frac{D\rho}{Dt} = 0$

4. Density variations ρ' are small compared to the background density ρ_0

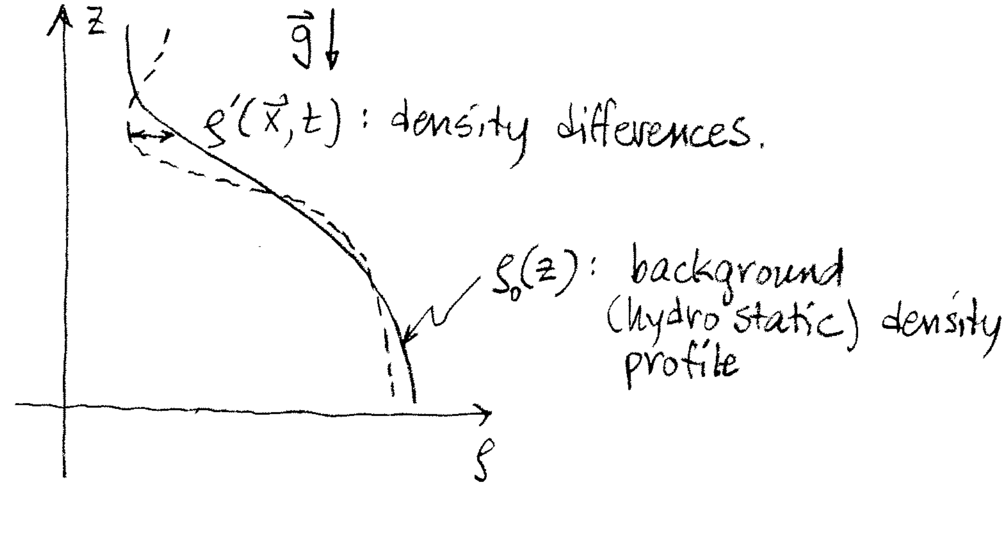
Justification:

2. & 3. are very good for non-breaking internal and interfacial waves. Must be modified when mixing occurs, but mixing is generally patchy and intermittent.

4. In the oceans $\rho_s \approx 1024$ kg/m³ (s ← saltwater). Freshwater has a density of $\rho_f \approx 1000$ kg/m³. Hence, $\rho' \approx \mathcal{O}(20)$ kg/m³ and $\rho_0 \approx 1000$ kg/m³:

$$\frac{\rho'}{\rho_0} \sim \mathcal{O}(10^{-2}) \ll 1 \checkmark$$

Notation:



Conservation of Mass:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

↓ product rule

$$\underbrace{\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i}} + \rho \frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i} = 0$$

$\frac{D\rho}{Dt}$ → 0: incompressible, non-mixing fluid

$$\nabla \cdot \vec{u} = 0$$

Conservation of Momentum:

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \rho f_i + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

ρf_i: body force.

μ: assumes constant; otherwise we have $\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]$.

Find hydrostatic solution:

x-dir: $0 = 0$; y-dir: $0 = 0$

z-dir:

$$0 = -\frac{\partial p_0}{\partial z} + \rho_0(-g) \rightarrow \frac{dp_0}{dz} = -g\rho_0$$

$$\begin{pmatrix} p_0 \\ \rho_0 \end{pmatrix} \rightarrow$$ hydrostatic solution, but $\rho_0 \neq$ const.

Modified Reynolds Decomposition:
flow occurs due to deviations (p' and ρ') from hydrostatic condition.

$$p(x_i,t) = p_0(x_i) + p'(x_i,t)$$
$$\rho(x_i,t) = \rho_0(x_i) + \rho'(x_i,t)$$

Substitute in dynamic equations:

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial(p_0+p')}{\partial x_i} + (\rho_0+\rho')f_i + \mu\frac{\partial^2 u_i}{\partial x_j^2}$$
$$= \frac{\partial p'}{\partial x_i} + \underbrace{\frac{\partial p_0}{\partial x_i} + \rho_0 f_i} + \rho' f_i + \mu\frac{\partial^2 u_i}{\partial x_j^2}$$

$= 0$ since p_0 and ρ_0 are solutions to hydrostatic equation.

Divide by ρ:

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \underbrace{\frac{\rho'}{\rho_0}f_i} + \nu\frac{\partial^2 u_i}{\partial x_j^2}$$

body force.

$$\frac{\rho'}{\rho_0}f_i \rightarrow -\frac{\rho' g}{\rho_0} = \underbrace{\frac{\rho_0-\rho}{\rho_0}g}$$

g' : reduced gravity.

Since $\frac{\rho'}{\rho_0} \ll 1$, the effect of stratification is to reduce the body force $\rightarrow$ dynamics like that in low gravity.

Finally:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + g' + \nu \frac{\partial^2 u_i}{\partial x_j^2}$$

Boussinesq Approximation:

Only density differences must retain knowledge of the true density if ρ' is small.

$$\rho(x_i,t) \approx \rho_o(x_i) \approx O(1000)\ \mathrm{kg/m^3}$$

$$\rho'(x_i,t) = \underbrace{\rho(x_i,t) - \rho_o(x_i,t)}_{O(0 \text{ to } 20)\ \mathrm{kg/m^3}}$$

$\therefore$ keep as unknown.

Hence, the governing equations become:

$$\nabla \cdot \vec{u} = 0$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\frac{1}{\rho_o}\nabla p' + \vec{g}' + \frac{\mu}{\rho_o}\nabla^2\vec{u}$$

Vorticity Equation:

Take curl of momentum equation:

$$\nabla \times \left\{ \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho} \nabla p + \vec{g}' + \nu \nabla^2 \vec{u} \right\}$$

$-\frac{1}{\rho}\nabla p$ generates two terms

$$-\nabla\left(\frac{1}{\rho}\right) \times \nabla p - \frac{1}{\rho} \nabla \times \nabla p$$

$-\nabla\left(\frac{1}{\rho}\right) \times \nabla p$: vorticity production; $\nabla \times \nabla p$: ϕ

Thus, we obtain:

$$\frac{D\vec{\omega}}{Dt} = \vec{\omega} \cdot \nabla \vec{u} + \nu \nabla^2 \vec{\omega} + \nabla p \times \nabla\left(\frac{1}{\rho}\right)$$

- $\vec{\omega} \cdot \nabla \vec{u}$: stretching
- $\nu \nabla^2 \vec{\omega}$: diffusion
- $\nabla p \times \nabla\left(\frac{1}{\rho}\right)$: production: Non-zero unless ∇p and $\nabla\left(\frac{1}{\rho}\right)$ are parallel.
  - ↳ Stratified flows are generally rotational.

Descriptive Parameters:

1.) $N = \left(-\frac{g}{\rho_0} \frac{d\rho}{dz} \right)^{-1/2}$ units of $\left[\frac{1}{T} \right]$.

↳ called buoyancy frequency.
Gives the natural frequency of oscillation for a fluid particle displaced from neutral buoyancy.

2.) $R_i = \frac{N^2}{(du/dz)^2} = -\frac{g}{\rho_0} \frac{d\rho/dz}{(du/dz)^2}$: $\frac{\text{restoring force of stratification}}{\text{destabilizing effect of shear velocity.}}$

↳ called gradient Richardson number.

Gives tendency of flow to mix due to shear flow → small values give mixing.

3.) $F_{r_d} = \frac{u}{\sqrt{g' h}}$

↳ called densimetric Froude number.
Gives phase speed of internal waves.